

LIGHT-CONE ZERO MODES REVISITED

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Abstract

The vacuum problem of light-cone quantum field theory is reanalysed from a functional-integral point of view.

1. INTRODUCTION

It has become a standard lore that “the light-cone (LC) vacuum is trivial”^a. This statement, however, appears somewhat paradoxical, as many important physical phenomena are usually attributed to the existence of a *nontrivial* vacuum. An incomplete list includes the notions of condensates, spontaneous symmetry breaking, vacuum tunneling (instantons) and decay, the Casimir energy and the cosmological constant. In order to clarify whether these features can be reconciled with a trivial vacuum, a necessary first step consists in precisely clarifying what is meant by the concept of a trivial vacuum. In this contribution we denote a vacuum as trivial if the associated vacuum persistence amplitude (VPA) is unity,

$$1 = \langle 0 | \hat{S} | 0 \rangle \equiv \exp(iW[0]) . \quad (1)$$

This is the probability amplitude that the vacuum state ‘persists’ in time, i.e. that the vacuum $|0\rangle$ in the remote past evolves to the vacuum in the distant future under the influence of the interaction encoded in the S-matrix \hat{S} . For a stable vacuum (no vacuum decay), this is a phase, $\exp(iW[0])$, and triviality of the vacuum means that the Schwinger functional vanishes, $W[0] = 0$. Note that we do not make a statement about the vacuum *state* $|0\rangle$ itself. Graphically, (1) can be represented as

$$1 = \exp \left\{ \begin{array}{c} \text{two circles} + \text{three circles} + \text{circle with horizontal line} + \dots \end{array} \right\} , \quad (2)$$

so that $W[0] = 0$ is the statement that all vacuum bubbles vanish. This is totally consistent with the absence of terms consisting of only creators or annihilators ($a^\dagger a^\dagger a^\dagger a^\dagger$, $aaaa$) in the LC Hamiltonian, which in turn implies that the interaction does not change the vacuum,

$$H_0 |0\rangle = 0 = H_{\text{int}} |0\rangle . \quad (3)$$

Writing the S-matrix in terms of the standard Dyson series, $\hat{S} = T^+ \exp(-i \int dx^+ H_{\text{int}})$, we are led back to (1). On the other hand, the VPA is alternatively given by the path integral

$$1 \stackrel{?}{=} \int \mathcal{D}\phi \exp iS[\phi] \equiv Z[0] \equiv \exp iW[0] , \quad (4)$$

where S denotes the classical action. Now, according to Feynman, the path integral constitutes a ‘space-time approach to quantum theory’ or, put differently, it is explicitly frame independent. The question mark in (4) represents the fact that in general the path integral is different from unity, $Z[0] \neq 1$, and this is an invariant statement which does not depend on the choice of coordinates (LC or else). This apparent paradox is easily resolved by recalling that the VPA or path integral is a mere normalization constant: in the generating functional $T[J]$ of Green functions it is just divided out leading to

$$T[0] \equiv Z[J]/Z[0]|_{J=0} = 1 , \quad (5)$$

^aSee e.g. the reviews [1] for extensive discussions of this issue.

which is unity by construction. The transition from Z to T can be viewed as a renormalization of the cosmological constant or, more practically, as the prescription to ‘omit the vacuum bubbles’. Using the exponentiation (4) of vacuum bubbles we may write

$$T[J] = \int \mathcal{D}\phi \exp i \{S[\phi, J] - W[0]\} , \quad (6)$$

from which we infer that the modified action $S' \equiv S - W[0]$ has a trivial vacuum by construction. This suggests that the ‘standard’ LC Hamiltonian (3) should correspond to the action S' rather than S . Of course, $W[0]$ in general is a divergent quantity so that, prior to renormalization, (6) is a somewhat formal statement. In what follows we will try to shed some light on the mechanism by which the subtraction of vacuum bubbles arises.

2. THE LC PATH INTEGRAL

While the path integral itself does not depend on the choice of coordinates, the integration measure as well as particular choices of boundary conditions and sources do^b. In this sense we may talk about a LC path integral. Consider now ϕ^4 -theory in d dimensions. The VPA in the presence of a source J is given by the functional integral

$$Z[J] = \int \mathcal{D}\phi \exp i \left\{ S_0[\phi] + \frac{\lambda}{4!}(\phi^2, \phi^2) + (\phi, J) \right\} \quad (7)$$

with the free action S_0 being the ‘quadratic form’ $S_0[\phi] \equiv -\frac{1}{2}(\phi, K_m \phi)$, defined in terms of the Klein-Gordon operator $K_m \equiv \square + m^2$, the inverse of which is the Feynman propagator Δ_m . It is instructive to have a look at its mixed Fourier representation in terms of LC time $x^+ = t + z$ and momenta k^+, \mathbf{k}_\perp ,

$$\Delta_m(x^+, k^+, \mathbf{k}_\perp) \equiv \begin{cases} \frac{\theta(k^+ x^+)}{i |k^+|} e^{-ix^+(\mathbf{k}_\perp^2 + m^2 - i\epsilon)/2k^+}, & k^+ \neq 0 \\ -\frac{2\delta(x^+)}{\mathbf{k}_\perp^2 + m^2 - i\epsilon}, & k^+ = 0, \end{cases} \quad (8)$$

which displays non-uniform behavior in the longitudinal momentum k^+ [3]. For $k^+ \neq 0$ one has a standard Feynman–Stückelberg interpretation. The *zero modes* (ZMs) with $k^+ = 0$, however, propagate instantaneously within the null-plane $x^+ = 0$. It is hence quite obvious that these modes will play a peculiar role in a Hamiltonian treatment describing the time evolution *off* this null-plane [4]. While the behavior (8) of the ZMs may seem strange in the first place, it is actually not entirely unfamiliar. Quite an analogous behavior shows up in the Coulomb gauge propagator,

$$D_{00}(x^0, \mathbf{k}) = -\frac{\delta(x^0)}{\mathbf{k}^2 - i\epsilon}, \quad (9)$$

which describes instantaneous ‘propagation’ of (virtual) scalar gauge bosons [5] resulting in the Coulomb interaction (see below). From (8) we are thus led to generally distinguish between field ZMs ω with $k^+ = 0$ and their complement φ which gives rise to an orthogonal decomposition, $\phi = \varphi + \omega$, $J = j + j$, $\mathcal{D}\phi = \mathcal{D}\varphi \mathcal{D}\omega$. If we have a look at the equations of motion stemming from S_0 (with sources present),

$$K_m \varphi = (\square + m^2) \varphi = j, \quad (10)$$

$$K_m^0 \omega \equiv (-\Delta_\perp + m^2) \omega = j, \quad (11)$$

we note that ω obeys a constraint which is analogous to Poisson’s equation for the temporal component of the photon field (in the presence of a static charge distribution ρ), $\Delta A_0 = \rho$. Hence, both A_0 and ω are

^bSee e.g. the nice discussion of the Casimir effect in [2]

‘non-dynamical’ fields (having no conjugate momenta) but are nevertheless integrated over in the path integral. However, while the A_0 -integration (both in the Abelian and non-Abelian case) is Gaussian, the ω -integration in ϕ^4 -theory,

$$Z[j, j] = \int \mathcal{D}\varphi \mathcal{D}\omega \exp i \{ S[\varphi, \omega] + (\varphi, j) + (\omega, j) \} , \quad (12)$$

is not, as the action

$$S[\varphi, \omega] = -\frac{1}{2}(\varphi, \mathbf{K}_m \varphi) - \frac{1}{2}(\omega, \mathbf{K}_m^0 \omega) + S_{\text{int}}[\varphi, \omega] \quad (13)$$

is quartic in ω . In order to have a simpler system with quadratic action we consider the coupling to external sources. We proceed by analogy with the Maxwell path integral in Coulomb gauge with static source ρ (or, equivalently, the vacuum expectation value of the Abelian Wilson loop). The latter is proportional to the integral

$$\int \mathcal{D}A_0 \exp i S_\rho[A_0] \sim \exp \{ -i T E[\rho] \} , \quad (14)$$

with quadratic action $S_\rho = -\frac{1}{2}(A_0, \Delta A_0) + (\rho, A_0)$ and yields the Coulomb potential (plus an irrelevant divergent self-energy that is independent of any length scales associated with the source),

$$E[\rho] = -\frac{(\rho, D_{00} \rho)}{2T} = \frac{1}{2} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{|\rho(\mathbf{k})|^2}{\mathbf{k}^2} = \frac{1}{4\pi r} + \text{const} , \quad (15)$$

where the last identity holds for two unit point charges at distance r in $d = 4$. Invoking covariance, the LC path integral with ZM source j should have the asymptotic behavior,

$$\int \mathcal{D}\omega \exp i S_j[\omega] \sim \exp \{ -i L P^+[j] \} , \quad (16)$$

where we assume a quadratic action $S_j = -\frac{1}{2}(\omega, \mathbf{K}_m^0 \omega) + (\omega, j)$. Performing the Gaussian integral (16) results in a source-generated longitudinal momentum proportional to a transverse Yukawa potential,

$$P^+[j] = \frac{(j, \Delta_m^0 j)}{2L} = -\frac{1}{4} \int \frac{dk^-}{2\pi} \int \frac{d^{d-2}k_\perp}{(2\pi)^{d-2}} \frac{|j(k^-, \mathbf{k}_\perp)|^2}{\mathbf{k}_\perp^2 + m^2} = -\frac{1}{2\pi} \delta(z^+) K_0(mz_\perp) + \text{const} . \quad (17)$$

Again, the last equality holds for two point sources of unit strength at distance (z^+, z_\perp) in $d = 4$. Furthermore, in the first equality, we have introduced the ZM propagator

$$\Delta_m^0(x - y) \equiv - \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik \cdot (x-y)}}{\mathbf{k}_\perp^2 + m^2} \equiv \overline{x} \text{ --- } \overline{y} , \quad (18)$$

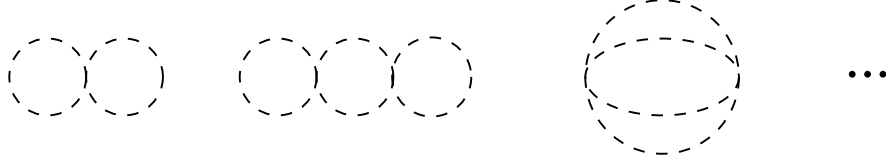
which will become important in a moment.

3. ZERO MODES

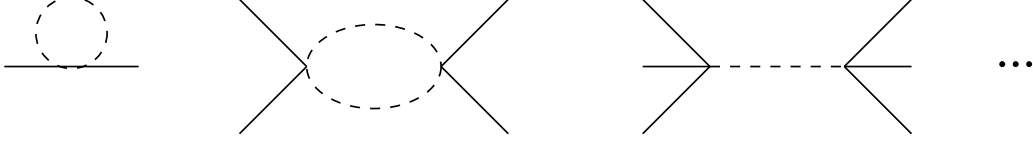
Let us go back to the more realistic situation of ϕ^4 -theory. There are two methods by which the zero modes can be integrated out approximately, perturbation theory and the saddle point approximation.

3.1 Perturbation theory

In this case one expects that the integration over ω corresponds to the limiting case of a Wilsonian momentum-shell integration over momenta from the interval $0 \leq k^+ \leq \delta$. This should give rise to an RG interpretation, which will be developed elsewhere. Here, we content ourselves with displaying the lowest-order effective interactions induced by the ω -integration. They typically include the ZM propagator Δ_m^0 , either within vacuum bubbles,



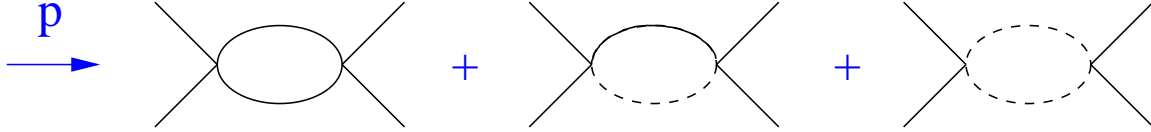
or in terms of new (nonlocal) φ -vertices,



On the other hand, solving the constraint equation,

$$\omega = \frac{\lambda}{3!} \frac{1}{2L} \int_{-L}^L dx^- \Delta_m^0 * (\varphi + \omega)^3 \quad (L \rightarrow \infty), \quad (19)$$

in perturbation theory [6] corresponds to ZM tree level as ZM loops are neglected [7]. In other words, the ZM sector is treated classically, i.e. to order \hbar^0 . For instance, if one considers the following 2-by-2 scattering amplitude,



one finds that only the first two diagrams are obtained by solving the constraint. They both vanish for $p^+ \rightarrow 0$ [7]. Only the last (ZM loop) diagram survives this limit. Unfortunately, this argument does not constitute a proof that ZMs cannot be neglected. In standard (noncovariant) LC perturbation theory the two time-orderings representing the first diagram have the correct covariant limit for $p^+ \rightarrow 0$ [8]. It would be interesting to study the critical behavior of ϕ^4 -theory using the new Feynman rules and compare with the results of [9] which were obtained by solving the constraint.

3.2 Semiclassical approximation

Alternatively, one may integrate out ω by expanding the action $S[\varphi, \omega]$ around the saddle point $\phi_0 = \omega_0 + \varphi_0 = \omega_0$, where we assume that ω_0 is a *constant* solution of the classical equation of motion (19). Again, the classical approximation neglects quantum fluctuations around ω_0 , and approximates the VPA by

$$Z[0] \simeq \int \mathcal{D}\varphi \exp iS[\varphi, \omega_0] = \int \mathcal{D}\varphi \int \mathcal{D}\omega \delta(\omega - \omega_0) \exp iS[\varphi, \omega]. \quad (20)$$

This is exactly the path integral which is obtained via the Dirac-Bergmann algorithm for constrained systems [10]. If one wants to go beyond (20), one has to calculate the fluctuation determinant. For constant background ω_0 this should yield the effective potential.

4. THE EFFECTIVE POTENTIAL

Indeed, the LC path integral (12) yields the standard formula $V_{\text{eff}} \equiv V_0 + V_1$, with $V_0[\omega_0]$ being the classical potential and $V_1[\omega_0]$ the one-loop effective potential,

$$V_1 \equiv -\frac{i}{2} \Omega_d^{-1} \text{Tr} \log (\Delta_m / \Delta_\mu). \quad (21)$$

Here, we have introduced the notation $\Omega_d \equiv \text{vol}(\mathbb{R}^d)$ and $\mu^2 \equiv m^2 + \lambda \omega_0^2 / 2$.

4.1 Calculation *with* ZMs

Expanding the log yields the 1PI diagrams

$$\Omega_d V_1 = \text{[Standard covariant tadpole]} + \text{[ZM loop]} \left\{ \text{[ZM bubble]} + \text{[ZM bubble with two vertices]} + \dots \right\}$$

where the first diagram is the standard covariant tadpole $T_m = \Omega_d^{-1} \text{Tr } \Delta_m$ of mass m , while the remaining ones are exclusively given by ZM loops. Each vertex attached to external ω -legs represents the factor $\lambda \omega_0^2/2$. The calculation of the tadpole T_m is done in standard dimensional regularisation (dimReg). The ZM bubbles are evaluated by means of the formula,

$$\int \frac{dk^-}{2\pi i} (k^+ k^- - M^2 + i\epsilon)^{-p} = \frac{(-1)^p}{p-1} \frac{\delta(k^+)}{(M^2)^{p-1}}, \quad p \neq 1, \quad (22)$$

which has first been used by Yan for $p = 2$ [11]. The k_\perp -integrations are done via dimReg. Altogether, we reproduce the standard dimReg result for the effective potential.

4.2 Calculation *without* ZMs

The idea is to relate the effective potential V_{eff} to the tadpole T_μ via differentiation with respect to the ZM-induced mass μ^2 ,

$$\frac{\partial V_1}{\partial \mu^2} = \frac{i}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - \mu^2 + i\epsilon} \equiv \frac{i}{2} T_\mu, \quad (23)$$

so that we are left with a single divergent tadpole integral. Intuitively, the tadpole can be viewed as the (infinite) volume of the mass-shell $k^2 = \mu^2$, see Fig. 1.

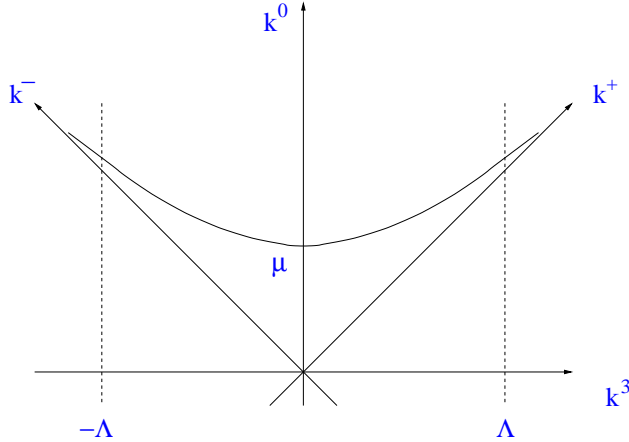


Fig. 1: Mass-shell and noncovariant cutoffs ($d = 2$).

It is obvious that a noncovariant cutoff $-\Lambda < |\mathbf{k}| < \Lambda$ renders the tadpole finite and gets translated in a cutoff for both large and small k^+ [12], $\mu^2/\Lambda < k^+ < \Lambda$, which eliminates any ZM contribution, and a transverse momentum cutoff [13], $k_\perp^2 < \Lambda^2 x(1-x) - \mu^2$ with $x \equiv k^+/\Lambda$. Employing this regularization, one obtains the standard cutoff result for V_{eff} in $d = 2$ and $d = 4$ [14].

5. SUMMARY AND OUTLOOK

Resuming it seems fair to say that the physics of LC ZMs still remains somewhat elusive. Nevertheless, the LC path integral provides (new) intuition. A trivial vacuum may be seen as resulting from the subtraction of a ‘cosmological constant’. After normal-ordering (which eliminates φ -loops), this constant must

be entirely due to ZM loops. These, however, are also zero if one sticks to ordinary constrained dynamics (or, equivalently, the classical ‘approximation’) in the ZM sector as has been common usage before. Effective ZM loops arise if one integrates out the ZM ω in the LC path integral in perturbation theory. A semiclassical approximation yields the standard effective potential. Details of the latter calculation show that summing the ZM contributions amounts to tuning the mass-dependent cutoff ($m^2 \rightarrow \mu^2$), which seems to suggest some RG interpretation. Of course, it would be desirable to make contact with a Hamiltonian formulation. We believe that the latter can only be defined with an intrinsic cutoff $k^+ > \delta$, making the vacuum trivial. While this removes ZMs, covariance presumably requires the presence of new interactions with $k^+ = 0$. The most interesting quantities one wants to know are the LC wave functions. Recently it has been shown that these can be extracted from the large-LC-time asymptotics of the LC Schrödinger functional [15]. The study of this functional is interesting by itself as it is one of the keys to Hamiltonian renormalization.

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